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Anomalous transport of cosmic ray electrons

B. R. Ragot[1,2] and J. G. Kirk[1]

[1] Max-Planck-Institut für Kernphysik, Postfach 10 39 80, D–69029 Heidelberg, Germany
[2] Max-Planck-Institut für Radioastronomie, Auf dem Hügel 69, D–53121 Bonn, Germany

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Abstract. Anomalous transport processes in which the variance of the distance travelled does not necessarily increase linearly with time: $\langle \Delta x^2 \rangle \propto t^\alpha$ with $0 < \alpha < 2$ are modelled using the formalism of continuous time random walks. We compute particle propagators which have the required dependence on space and time and use these to find the spatial dependence of the synchrotron radiation emitted by a population of continuously injected electrons. As the electrons are transported away from the source they cool, and the synchrotron spectrum softens. Subdiffusive ($\alpha < 1$) transport – corresponding to stochastic trapping, or restriction of the transport across the average direction of a stochastic magnetic field – produces a much slower rate of change of spectral index than does supra-diffusion ($1 < \alpha < 2$) – which occurs when particles move almost without scattering, in a field containing large ordered regions. Application to the diffuse emission of the outer parts of the Coma cluster favours an interpretation in terms of supra-diffusion.

Key words: diffusion – magnetic fields – plasmas – turbulence – galaxies: clusters: Coma; radio continuum: galaxies

1. Introduction

Many of the properties of cosmic rays can be understood on the basis of a propagation model in which they execute isotropic diffusion between injection at sources distributed in the galactic disk and escape at certain boundaries whilst possibly being carried out of the galaxy in a wind (Lerche & Schlickeiser 1982 Berezinskii et al. 1990, Webber et al. 1992, Bloemen et al. 1993). On the other hand, it has been known for many years that these particles are 'magnetised' in the sense that their Larmor radius is much smaller than the mean free path between 'collisions' implied by the diffusion coefficient. This means that particles diffuse primarily along the magnetic field, and only very slowly across it. Since the galactic magnetic field lies predominantly in the galactic plane (e.g., Zweibel & Heiles 1997), it is surprising that isotropic diffusion describes cosmic ray propagation so well. One might instead expect cosmic rays to be transported anomalously slowly in the direction normal to the disk (Getmantsev 1963). The success of the simple diffusion model of propagation (or even of the more primitive but related 'leaky-box' model) appears to show that this is not the case, at least for the bulk of cosmic rays. This is presumably due to fluctuations in the magnetic field, possibly connected with instabilities, which cause a given field line to wander out of the galactic plane, and give rise to an effective diffusion across the direction of the mean magnetic field (Jokipii & Parker 1969, Jokipii 1973).

Nevertheless, it may not always be the case that the transport process can be described by an effective diffusion coefficient. Since its first mention by Getmantsev (1963), anomalous transport, i.e., transport in which the mean square deviation of a particle from its position at a given time $\langle \Delta \mathbf{x}^2 \rangle$ does not depend linearly on the elapsed time, has been discussed in two astrophysical contexts. Firstly, Chuvilgin & Ptuskin (1993) derived a kinetic equation for cosmic ray transport for the case in which $\langle \Delta \mathbf{x}^2 \rangle \propto t^{1/2}$, but pointed out that this type of transport can be expected only when considering timescales short compared to the time taken for a particle to effectively decorrelate from a given magnetic field line. Many effects such as particle drifts, temporal changes in the magnetic field or just the chaotic structure of the field itself can cause decorrelation, but although it is very difficult to estimate this time in an astrophysical situation, the success of the diffusion model suggests that cosmic rays do indeed decorrelate from the field faster than they escape from the galaxy (which takes about 10^8 years).

The second astrophysical context is that of particle acceleration at a shock front. In this case, the natural timescale is the acceleration time, which is thought to be a strong function of energy, and to vary over several orders of magnitude for those cosmic rays accelerated at supernova shocks (Dendy et al 1995, Duffy et al. 1995). Here too, it

Send offprint requests to: J.G. Kirk

is difficult to make a realistic estimate of the decorrelation time of a particle from the magnetic field. However, predictions can be made of the spectrum and spatial distribution of those particles undergoing anomalous transport, which might enable them to be distinguished from diffusing particles (Kirk et al. 1996a).

A problem which is similar in many respects arises in the confinement of plasma in fusion devices, and in this context there has been considerable interest in recent years in anomalous, non-diffusive transport models (Rechester & Rosenbluth 1978, Kadomtsev & Pogutse 1979, Isichenko 1991a,1991b, Rax & White 1992, Wang et al. 1995). In particular, use of the formalism developed for continuous time random walks (CTRW – see Montroll & Weiss 1965) has been advanced (Balescu 1995). This approach is widely used in the problem of anomalous transport in random media (e.g., Bouchaud & Georges 1990, Shlesinger et al. 1993). In this paper, we discuss these techniques and show how they relate to the work already done on anomalous transport in astrophysics. We point out that they offer a more general approach to the problem, enabling one to relax the rather restrictive assumptions concerning the statistical properties of the magnetic field employed hitherto. As an example, we examine a new astrophysical application – that of the synchrotron emission of a population of relativistic electrons which lose energy whilst being transported away from the site of their acceleration. Both the spatial extent of the radiating electrons and the observed spectrum depend on the nature of the transport process. These calculations are especially relevant to the interpretation of high resolution radio observations in several astrophysical situations, including spiral galaxies seen 'edge-on' (e.g., Hummel et al. 1991) as well the diffuse emission from clusters of galaxies (Schlickeiser et al. 1987, Kirk et al. 1996b).

2. Anomalous transport

In a regular magnetic field topology, the transport of charged particles across the field is due to the collisions of the particles and their finite Larmor radii. However, a perturbation of the magnetic field results in a wandering of the field lines and a potentially much faster transport of the particles. This transport is widely attributed to the effect of the large-scale random field component, which would produce, following the quasilinear theory, a diffusion of the field lines across the direction of the the average field (Jokipii & Parker 1969, Kadomtsev & Pogutse 1979). In fusion applications, it is generally assumed that particle collisions lead to diffusive transport along each individual field line. In astrophysical applications, where collisions can be neglected, it is usual to assume that small scale fluctuations in the magnetic field play this role, so that here too, particles diffuse along field lines (Chuvilgin & Ptuskin 1993). As long as the particles remain correlated to a given patch of field lines, the combination of these two diffusions results in sub-diffusion of the particles i.e., $\langle \Delta x^2(t) \rangle \propto t^\alpha$, with $\alpha = 1/2$, as described by Getmantsev (1963). (In the general case, sub-diffusion refers to all such processes when $0 < \alpha < 1$.) However, the exponential divergence of neighbouring field lines and the resulting stretching of a field line patch lead to decorrelation of a particle from its field line, and thus to the large-scale diffusion of the particles, as pointed out by Rechester and Rosenbluth (1978), with a diffusion coefficient given by the expression:

$$D_{RR} = D_{st} 2\kappa_\parallel / l_{c\delta} \tag{1}$$

where $l_{c\delta} = l_c \log(1/k_0\delta)$, $\delta = l_c\sqrt{\kappa_\perp/\kappa_\parallel}$, where l_c is the exponentiation length of the field lines. Here D_{st} is the quasilinear diffusion coefficient of the field lines, which has the dimensions of a length, k_0 is a characteristic wave number of the magnetic turbulence, $\kappa_\parallel$ and $\kappa_\perp$ are the quasilinear diffusion coefficients of the particles along and across the direction of the local magnetic field, respectively. The transport of particles across a turbulent magnetic field is thus sub-diffusive for short timescales, but crosses over to normal diffusion in the limit of the long timescales. In the following, we investigate the anomalous transport regime, which applies if the natural timescale of a particular problem – such as escape from the galaxy, acceleration at a shock, or loss of energy by synchrotron radiation, is shorter than the decorrelation time.

In a statistically homogeneous medium, the density of particles, $n(\boldsymbol{x}, t)$, is related to the source $Q(\boldsymbol{x}', t')$ by the propagator P:

$$n(\boldsymbol{x}, t) = \int \mathrm{d}\boldsymbol{x}' \int \mathrm{d}t'\, P(\boldsymbol{x} - \boldsymbol{x}', t - t') Q(\boldsymbol{x}', t') \,. \tag{2}$$

For diffusive transport, P typically contains a factor $\exp(-|\boldsymbol{x}|^2/Dt)$, and is the Green's function of the diffusion equation.

For sub-diffusive transport with $\alpha = 1/2$, Rax & White (1992) have determined the propagator by combining two diffusive propagators using a Wiener integration method. They considered the transport in cylindrical symmetry across the z direction (i.e., a 2-dimensional problem, with $|z|$ playing the role of time), and obtained:

$$P(r, t) = \frac{H(t)}{2\pi \left(4r^2 D_{\mathrm{M}}^2 \kappa_\parallel t\right)^{1/3}} \exp\left[\frac{-3r^{4/3}}{4\left(4D_{\mathrm{M}}^2 \kappa_\parallel t\right)^{1/3}}\right] \tag{3}$$

where r is the radius in cylindrical coordinates, and $H(t)$ the Heaviside function. Duffy et al. (1995) considered the transport perpendicular to a plane shock front. In a cartesian coordinate system with the shock in the y–z plane, and assuming no gradients in the y direction are present, they obtained the propagator

$$P(x, t) = \int \mathrm{d}z\, \frac{\exp\left[-x^2/\left(4D_{\mathrm{M}}|z|\right) - z^2/\left(4\kappa_\parallel t\right)\right]}{\sqrt{4\pi D_{\mathrm{M}}|z|}\sqrt{4\pi\kappa_\parallel t}} \tag{4}$$

Approximating the integral using the method of steepest descents, they found

$$P(x,t) \;\approx\; \eta(D_{\mathrm{M}}\kappa_{\|}^{1/2}|x|t^{1/2})^{-1/3}\exp\left(\frac{-\beta|x|^{4/3}}{D_{\mathrm{M}}^{2/3}\kappa_{\|}^{1/3}t^{1/3}}\right) \quad (5)$$

with $\eta = 2^{1/3}(3\pi)^{1/2}$ and $\beta = 3/2^{8/3}$. For this 1-dimensional problem of $\alpha = 1/2$ sub-diffusion, Chuvilgin & Ptuskin (1993) derived an equation describing the evolution of the particle density (their Eq. B.12) and solved it to find the above propagator. Essentially the same equation was also found by Balescu (1995). It can be written:

$$\partial_t n(x,t) \;=\; D_0\Bigg[\nabla_x^2 n(x,t) - \frac{1}{2\sqrt{\pi}\tau_D}\int_{1/\pi}^{t} d\tau\left(\frac{\tau_D}{\tau}\right)^{3/2}\nabla_x^2 n(x,t-\tau)\Bigg] \quad (6)$$

with $D_0 = \sigma^2/2\tau_D$. This is a non-Markovian diffusion equation: the integral term is characteristic of the long-time memory of the dynamics.

In a realistic situation, the topology of the magnetic field may be more complicated than just a pure stochastic sea with its associated diffusion of the field lines. In fusion plasmas, for example, there can exist ordered structures, 'stability islands', in which particles can be trapped for long periods of time, (referred to as 'sticking'), leading to sub-diffusive large-scale transport of the field lines themselves (e.g., White et al. 1993). On the other hand, the field lines may also wander faster than implied by diffusion. In this case, the field lines are said to perform 'flights', during which they maintain almost the same direction for a relatively long distance. Ultimately, the large-scale transport of field lines is the result of competition between 'sticking' and 'flights', and can yield transport regimes of many different kinds, ranging from slow sub-diffusion (almost perfect sticking) to fast supra-diffusion (domination of flights). In terms of α, the physically relevant range is $0 < \alpha < 2$, with $\alpha = 2$ corresponding to transport completely dominated by a single straight-line flight.

To find the particle propagator, it is necessary to combine the propagator for field lines with that for particle motion along the field, as described above for the case of $\alpha = \frac{1}{2}$. Here too, the assumption of diffusive transport can be generalised. Particles may, in fact, undergo no scattering at all, in which case they propagate ballistically along the field lines, as assumed by Achterberg & Ball (1994). On the other hand, they may be trapped between magnetic mirrors on a segment of a field line. Once again, the transport can be characterised by an index α which lies between 0 and 2. However, as we show below, the combination of two propagators does not extend the overall range of α which is permitted. In fact, the effect of superimposing transport with the two indices α_1 and α_2 is described by a single value $\alpha = \alpha_1\alpha_2/2$.

3. Continuous time random walks

We do not aim to describe the microscopic dynamics of particles moving in a turbulent magnetic field in a realistic way, but rather look for a model which is able to reproduce their transport characteristics and gives the density profile. In terms of the propagator $P(\boldsymbol{x},t)$, the density is given by Eq. (2), once the rate $Q(\boldsymbol{x},t)$ at which particles are injected is specified. This can then be used to compute, for example, the synchrotron radiation from electrons injected at a plane surface (such as a spiral galaxy) and subsequently transported outwards.

The method we adopt to model P is that of continuous time random walks (CTRW). In this, the transport of a particle is modelled as a succession of instantaneous jumps of arbitrary length, separated by pauses of arbitrary duration. Each of these steps is independent of the previous steps. We denote by $f(\boldsymbol{x})$ the probability distribution function (PDF) of a jump described by $\boldsymbol{x}$, and by $\psi(t)$ the PDF of a pause of duration t, and use their Fourier and Laplace transforms:

$$\tilde{f}(\boldsymbol{k}) = \int d\boldsymbol{x}\, f(\boldsymbol{x})e^{ik.x} \quad ; \quad \hat{\psi}(s) = \int_0^{+\infty} dt\,\psi(t)e^{-st} \ . \quad (7)$$

The type of anomalous transport characterised by α and discussed above arises from the asymptotic behaviour of $P(\boldsymbol{x},t)$ as $|\boldsymbol{x}| \to +\infty$ and $t \to +\infty$. This corresponds to the $|\boldsymbol{k}| \to 0$, $s \to 0$ asymptotic behaviour of $\hat{\tilde{P}}(\boldsymbol{k},s)$, for which the expansions of $\hat{\psi}(s)$ and $\tilde{f}(\boldsymbol{k})$ around 0 are required. To keep the treatment general, we consider first the motion of a particle in a space of dimension d. The functions $\hat{\psi}(s)$ and $\tilde{f}(\boldsymbol{k})$ are then prescribed as

$$\hat{\psi}(s) \;=\; 1 - \tau_D^\alpha s^\alpha \ , \quad s \to 0 \quad (8)$$

$$\tilde{f}(\boldsymbol{k}) \;=\; 1 - \frac{1}{2d}\sigma^\beta|\boldsymbol{k}|^\beta \ , \quad |\boldsymbol{k}| \to 0 \ , \quad (9)$$

with $0 < \alpha, \beta \le 2$, and τ_D, σ positive constants. In Eq. (9), we have assumed that $\tilde{f}$ depends only on $|\boldsymbol{k}|$, i.e., that there is no preferred direction for a particle jump. The particular values $\alpha = 1$, $\beta = 2$ yield a diffusive process with a Gaussian propagator, whereas the case $\beta < 2$, leads to a random walk with an infinite second moment $\langle x^2\rangle$, corresponding to Lévy flights.

In a classical paper, Montroll and Weiss (1965) have shown that the Fourier-Laplace transform of $P(\boldsymbol{x},t)$ has the following form

$$\hat{\tilde{P}}(\boldsymbol{k},s) = \frac{1-\hat{\psi}(s)}{s[1-\hat{\psi}(s)\tilde{f}(\boldsymbol{k})]} \ . \quad (10)$$

Inserting the expansions Eq. (9) we find that for large $|\boldsymbol{x}|$ and t

$$\langle|\boldsymbol{x}|^2\rangle \;=\; \int_{-\infty}^{+\infty} d\boldsymbol{x}\,|\boldsymbol{x}|^2 P(\boldsymbol{x},t)$$

$$= \frac{1}{2\pi i}\frac{1}{2\pi}\int_\Gamma \mathrm{d}s \int_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{k} \int_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{x}$$
$$\exp\left(st - i\boldsymbol{k}.\boldsymbol{x}\right)|\boldsymbol{x}|^2 \frac{\tau_D^\alpha s^{\alpha-1}}{\tau_D^\alpha s^\alpha + \sigma^\beta |\boldsymbol{k}|^\beta/(2d)}$$
$$\propto t^{2\alpha/\beta} \qquad (11)$$

and the transport coefficient is $\mu = 2\alpha/\beta$. In the following, we find it convenient for the inversion of the Fourier transforms to choose $\beta = 2$, and describe all transport regimes using the parameter range $0 < \alpha \leq 2$:

$$\hat{\psi}(s) \;=\; 1 - \tau_D^\alpha s^\alpha \;,\;\; 0 < \alpha \leq 2 \qquad (12)$$
$$\tilde{f}(\boldsymbol{k}) \;=\; 1 - \frac{1}{2d}\sigma^2 |\boldsymbol{k}|^2 \;, \qquad (13)$$

for small s and $|\boldsymbol{k}|$. The choice $\alpha = 1$ has the same asymptotic behaviour as the function $\hat{\psi} = \exp(-s/\tau_D)$, i.e., $\psi = \delta(t - \tau_D)$, which corresponds to scattering after a fixed time interval, yielding diffusion. However, for $\alpha \neq 1{,}2$, we have

$$\psi(t) \;=\; \frac{1}{\tau_D}\frac{\alpha}{\Gamma(1-\alpha)}\left(\frac{t}{\tau_D}\right)^{-1-\alpha} \qquad (14)$$

for $t \to +\infty$. This is a long-tailed distribution function. Its slow decrease at large t leads to the same kind of memory effect built in to the non-Markovian transport equation (6) constructed by Chuvilgin & Ptuskin (1993) and Balescu (1995) for the case $\alpha < 1$. We do not consider $\alpha > 2$, since in this case the quantity $\left\langle \boldsymbol{x}^2 \right\rangle / t^2$ increases without limit for large t, and would at some stage exceed the actual physical particle speed.

From the expansions (13) the asymptotic behaviour of the propagator at large $|\boldsymbol{x}|$ and t can be found by an inverse Fourier transformation followed by an inverse Laplace transformation, which is performed approximately using the method of steepest descents. The resulting expression can be used for all values of $\boldsymbol{x}$ and t once the normalisation has been corrected. It represents a convenient one-parameter model of particle transport which exhibits the anomalous behaviour in which we are interested. The details of the derivation are given in Appendix A. Here we simply present the results, expressed in terms of a dimensionless similarity variable defined according to

$$\xi \;=\; \sqrt{2d}\frac{|\boldsymbol{x}|}{\sigma}\left(\frac{\tau_D}{t}\right)^{\alpha/2} \qquad (15)$$

Using this variable, the propagator separates:

$$P_d(\boldsymbol{x}, t) \;=\; \sigma^{-d}\left(\frac{\tau_D}{t}\right)^{\alpha d/2} \Xi_d(\xi) \;, \qquad (16)$$

and we find

$$\Xi_d(\xi) \;=\; C_d \xi^{-d(1-\alpha)/(2-\alpha)} \exp\left[-A\xi^{2/(2-\alpha)}\right] \;, \qquad (17)$$

where the (positive) constants C_d and A are given in the appendix (Eqs. A3 and A4). These propagators are normalised such that particle number is conserved: $\int \mathrm{d}^d\boldsymbol{x} P(\boldsymbol{x}, t) = 1$. They display at all t the behaviour required of the asymptotic dependence of the variance in anomalous transport. In particular, they possess for all t the property

$$\langle x^\nu \rangle \propto t^{\nu\alpha/2} \;. \qquad (18)$$

For one-dimensional propagation ($d = 1$), which describes, for example, the transport of particles in x away from a uniform source in the y–z plane, we have for the standard diffusive case ($\alpha = 1$) the well-known form

$$P(x, t) \;=\; \frac{1}{\sqrt{4\pi D t}} \exp\left[-\xi^2/4\right] \qquad (19)$$

with $D = \sigma^2/(2\tau_D)$ and $\xi = |x|/\sqrt{Dt}$. In the $\alpha = 1/2$ sub-diffusive case, which can be considered as a double diffusion, we find:

$$P(x, t) \;=\; \frac{1}{\sigma\sqrt{6\pi}}\left(\frac{\tau_D}{t}\right)^{1/4} \xi^{-1/3} \exp\left[-3\left(\xi/4\right)^{4/3}\right] \;, \qquad (20)$$

which agrees with the propagator for $d = 1$ given in Eq. (5) and derived by Duffy et al. (1995), with $\sigma = (2D_M)^{2/3} k_\parallel^{1/3}/u^{1/3}$ and $\tau_D = (2D_M)^{2/3} k_\parallel^{1/3}/u^{4/3}$. (Similarly, the case $d = 2$ reproduces the propagator (3) of Rax & White 1992). In analogy with the formulation of Eq. (4), it is possible to consider other values of α as arising from a convolution of a propagator describing motion along the field line together with one describing the wandering of the field. Using the method of steepest descents, it is straightforward to show that $P_{d,\alpha}$ has the following convolution property:

$$\int_{-\infty}^{+\infty} ds\, P_{d,\mu_1}(\boldsymbol{x}, |s|)\, P_{1,\mu_2}(s, t) = P_{d,\mu_1\mu_2/2}(\boldsymbol{x}, t) \;, \qquad (21)$$

(note that the normalisation is preserved) so that the combined transport process can always be described using a value of α between 0 and 2.

To illustrate the properties of these propagators, they are plotted for three different values of α in Fig. (1). The diffusive propagator is simply a gaussian curve given by Eq. (19) according to which the probability of finding a particle at a particular distance drops off smoothly with increasing distance from the point (plane) of injection. The sub-diffusive propagator, on the other hand, falls off much more steeply initially (it has an integrable singularity at the point of injection), but at larger distances decreases more slowly than diffusion. Acting on an ensemble of particles, sub-diffusive transport tends to confine some of them close to the injection point, but propagates others very rapidly to large distance. This is a result of the larger spread in position of sub-diffusing particles of a given age,

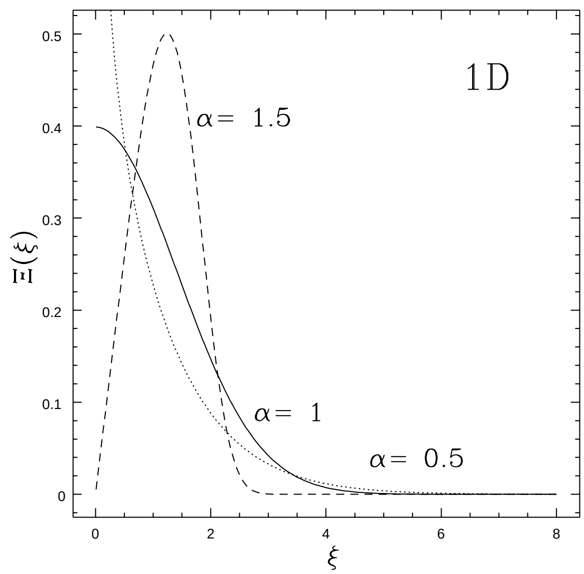

Fig. 1. The dimensionless propagator (Eq. 16) for one-dimensional transport in three different regimes: standard diffusion ($\alpha = 1$, solid line), sub-diffusion ($\alpha = 1/2$ dotted line) and supra-diffusion ($\alpha = 3/2$ dashed line), as a function of the dimensionless similarity variable ξ, which, for fixed time, is proportional to distance (see the definition in Eq. 15).

compared with diffusing ones. Thus, despite the fact that sub-diffusion produces, on average, slower transport, a minority of particles experiences comparatively rapid transport. Supra-diffusion displays the opposite behaviour and is similar to ballistic transport, or propagation at constant speed without scattering. The propagator is strongly peaked around the value $\xi \approx 1$. Very few particles are to be found lingering close to the point of injection, and very few escape to large distance.

Equation (17) is the basic result of this section. We now turn to the question of the observable consequences of these propagators.

4. Synchrotron radiation from anomalously transported electrons

The synchrotron radiation emitted by relativistic electrons is the most important diagnostic available for the study of the transport of these particles in astrophysical environments. In this section we compute the spatial dependence of the surface brightness and of the spectral index which is to be expected when the different kinds of particle transport discussed above dominate.

The energy losses suffered by a relativistic particle emitting synchrotron radiation, or undergoing inverse compton scattering whilst propagating through a homogeneous magnetic field, or a homogeneous soft radiation field can be described by the equation

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = -g\gamma^2 \tag{22}$$

where γ is the Lorentz factor of the particle and g a factor which depends on the magnetic field strength, and the energy density of the photon field. In units suited to the application to clusters of galaxies (see below) one has

$$g = 4.1 \times 10^{-14} \left(B_{\mu\mathrm{G}}^2 + 12.3\right) \qquad \mathrm{yr}^{-1} \tag{23}$$

where the photon field is assumed to be that of the cosmic microwave background, which is for this purpose equivalent to a magnetic field strength of $3.5\,\mu$G. Defining the differential rate dQ at which electrons are injected into the system at time t as $\mathrm{d}Q = Q(\boldsymbol{x}, \gamma, t)\mathrm{d}^d\boldsymbol{x}\mathrm{d}\gamma$, the resulting electron density, allowing for energy losses, is

$$n(\boldsymbol{x}, \gamma, t) = \int_\gamma^\infty \mathrm{d}\gamma' \int \mathrm{d}^d\boldsymbol{x}' \int_{-\infty}^t \mathrm{d}t' P(\boldsymbol{x} - \boldsymbol{x}', t - t') \\ Q(\boldsymbol{x}', \gamma', t')\delta\left(\gamma - \frac{\gamma'}{1 + g\gamma'(t - t')}\right) \ . \tag{24}$$

Consider electrons continuously injected with a power-law distribution in γ at the point $\boldsymbol{x} = 0$, i.e.,

$$Q(\boldsymbol{x}, \gamma, t) = Q_0\gamma^{-p}\delta(\boldsymbol{x}) \ , \tag{25}$$

where Q_0 is a constant. At all points and at all energies of interest the distribution has had sufficient time to reach a stationary state, which is given by the integral

$$n(\boldsymbol{x}, \gamma) = Q_0\gamma^{-p} \int_0^{1/(g\gamma)} \mathrm{d}t' P(\boldsymbol{x}, t')(1 - g\gamma t')^{p-2} \ . \tag{26}$$

In order to compute the surface brightness on the sky of the synchrotron radiation from these electrons, two more integrations are required – firstly over the line of sight through the source and secondly over γ, after multiplying n by the synchrotron kernel. Furthermore, information about the magnitude and direction of the magnetic field along the line of sight is needed. For our purpose, it is sufficient to use the delta-function or 'monochromatic' approximation to the synchrotron kernel (see, for example, Mastichiadis & Kirk 1995) and we will simplify the discussion by using a constant, angle-averaged emissivity in the line of sight integral (see, however, Crusius & Schlickeiser 1988). As a result, the surface brightness at a given frequency is simply proportional to $\int \mathrm{d}z n(\boldsymbol{x}, \gamma)$, where z is measured along the line of sight and where γ is fixed, being proportional to the square root of the observing frequency.

In the one-dimensional case in which electrons are injected in a plane, and the line of sight is parallel to this plane, displaced by a distance x, the resulting surface brightness is directly proportional to $n(x, \gamma)$ given by Eq. (26). This expression is simple to evaluate numerically. For various special cases, such as diffusion ($\alpha = 1$),

or injection with $p = 2$ the integral can be written in closed form (see Appendix B for details). The resulting profile is shown in Fig. 2 for the three qualitatively different types of transport considered in Sect. 3. It is important

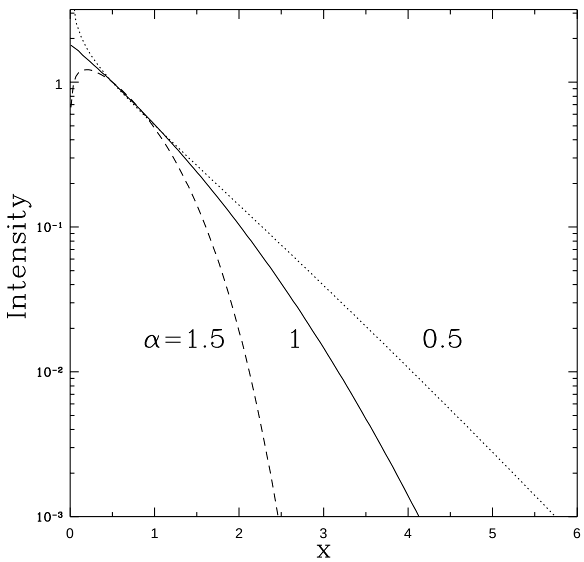

Fig. 2. The surface brightness of synchrotron radiation from a planar source of relativistic electrons, seen edge-on, as a function of distance x from the plane of injection, normalised to unity at $x = 0.5$. Distance is expressed in units of the scale length x_s (Eq. 27). Electrons are injected at $x = 0$ with a spectrum $Q \propto \gamma^{-2.5}$

to note that the shapes of these profiles are characteristic of the type of transport involved and are independent of the frequency of observation. The assumptions entering the calculation are that the magnetic field strength is constant, that the source has the stated geometry, and that the electron injection has been steady over a sufficiently long period of time. The abscissa in Fig. 2 is the scaled radius $\hat{x} = x/x_s$, where

$$x_s \; = \; \frac{\sigma}{\sqrt{2}(g\gamma\tau_D)^{\alpha/2}} \;\; , \tag{27}$$

which is in general a function of observing frequency. Sub-diffusive transport ($\alpha < 1$), which tends to trap particles close to the point of injection, displays an integrable singularity at $x = 0$, whereas supra-diffusive transport, which quickly moves particles away from the source, initially increases at small x. The brightness profile for diffusive transport is monotonically decreasing with x. In Fig. 2 the surface brightness is arbitrarily normalised to unity at $x = 0.5$. In a realistic situation, in which the electron source is distributed in space, it may not be possible to detect these features. In fact, the smooth almost exponential decrease of surface brightness with distance means that it would be difficult if not impossible to distinguish observationally between the different types of transport from these profiles alone.

A more promising approach lies in the measurement of synchrotron spectra as a function of position. In the delta-function approximation, the synchrotron spectral index $\alpha_{\rm s}$ is simply related to the γ-dependence of the electron density integrated along the line of sight. To compute this, one must take account of the dependence of the scaling radius on γ. Defining the frequency dependence of the surface brightness by $I_\nu \propto \nu^{\alpha_{\rm s}}$, we have

$$\alpha_{\rm s} \; = \; -\frac{1}{2}\left\{p - \frac{(\alpha)}{2}\left[1 + \left(\frac{\partial \log(N)}{\partial \log(\hat{x})}\right)\right]\right\} \tag{28}$$

where $N = \int \mathrm{d}z \, n(\boldsymbol{x}, \gamma)$. The resulting spectral index as a function of x is plotted in Fig. 3 for $p = 2.5$.

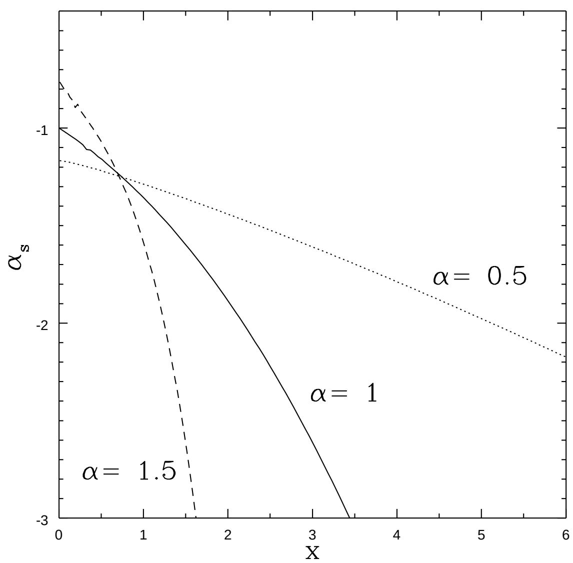

Fig. 3. The spectral index of synchrotron emission as a function of (scaled) radius. The dotted line corresponds to sub-diffusion with $\alpha = 0.5$, the solid line to diffusion ($\alpha = 1$) and the dashed line to supra-diffusion ($\alpha = 1.5$). Electrons are injected continuously in the plane $x = 0$ with a spectrum $Q \propto \gamma^{-2.5}$. The total emission integrated over x is thus a pure power-law of index $\alpha_{\rm s} = -1.25$.

This figure shows that the different types of transport produce very different spectra as the particles propagate away from the source. Close to the plane of injection, supra-diffusion gives approximately the "uncooled" value of the spectral index $\alpha_{\rm s} = -0.75$. This is because only recently injected particles remain close to $x = 0$ – older particles have a very small chance of return in the supra-diffusive case. As α decreases, more and more older particles are confined close to the point of injection, so that the spectrum for diffusive and sub-diffusive transport is softer.

In general, the age distribution of particles produced by each transport process determines the spectrum. Thus, supra-diffusion ($\alpha = 1.5$, dashed line) has a very narrow spread of particle age at a given x. Once the distance exceeds x_s, the spectrum becomes very soft, since the number of radiating particles decreases rapidly. In the case of diffusion ($\alpha = 1$, solid line) the age distribution is gaussian and produces a more gradual softening, whereas the sub-diffusive case ($\alpha = 0.5$, dotted line) not only traps some particles close to $x = 0$ but, also transports a few particles out rapidly outwards. Thus, the spectrum remains rather hard even at large radius. An observationally relevant measure of this dependence is the change in spectral index over the distance required for the brightness to drop by one order of magnitude from its value at $x = 0.5$. For supra-diffusion, the spectrum softens by 1.8 from -1 at $x = -.5$ to -2.9 at $x = 1.6$; for diffusion it softens by 0.8 from -1.1 at $x = 0.5$ to -1.9 at $x = 2$ and for sub-diffusion we find a change of only 0.3 from -1.2 at $x = 0.5$ to -1.5 at $x = 2.2$. Thus, a relatively slow change in synchrotron spectral index indicates sub-diffusive behaviour.

As an example, we consider the diffuse radio emission observed from the Coma cluster of galaxies. There is active debate as to the origin of the relativistic electrons responsible for this emission. It appears that simple diffusion from one of the galaxies near the centre is not, by itself, a viable explanation (e.g., Kirk et al. 1996b) and that some kind of distributed injection and/or acceleration is necessary (Schlickeiser et al. 1987). However, observations by Giovannini et al. (1993) have shown that there is a distinction between the central parts of the cluster (say within a radius of 500 kpc) and the outer parts (in this paper we adopt the value $H_0 = 50\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, so that 1 sec of arc is equivalent to 700 pc). If we assume that the processes of injection and acceleration occur within a sphere of radius 500 kpc, we can apply the above propagators to model the surface brightness and spectral index of the emission in the outer parts.

Figure 4 shows the observed emission at 430 MHz, as presented by Valtaoja (1984), from the data of Hanisch (1980). The horizontal "error bars" in this representation are an expression of spread in size of the emission when measured along the E-W and N-S directions. Superimposed on the data are three models computed using the one-dimensional propagators. Clearly, the (almost) spherical geometry of the cluster is important for $r > 1000$ kpc. However, the model should represent the data fairly well within the range $500 < r < 1000$. It is apparent from this figure that each of the three types of transport can produce the required fall off with distance, but that the associated predicted softening in the spectral index, which is also shown in the figure, is quite different in each case. The spectral index maps of Giovannini et al. (1993), which were computed from the ratio of the fluxes at 326 and 1380 MHz, show that within a few hundred kpc of the edge (at 500 kpc) the spectrum softens rapidly – by more than 0.6 in 200 kpc. This is indicated by an upper limit in the figure. Of the curves presented in Fig. 4, only that corresponding to supra-diffusion produces a comparable effect. Thus, in the stationary state, and assuming *in situ* acceleration is negligible, diffusion is not capable of propagating the electrons outwards from the inner 500 kpc of the cluster.

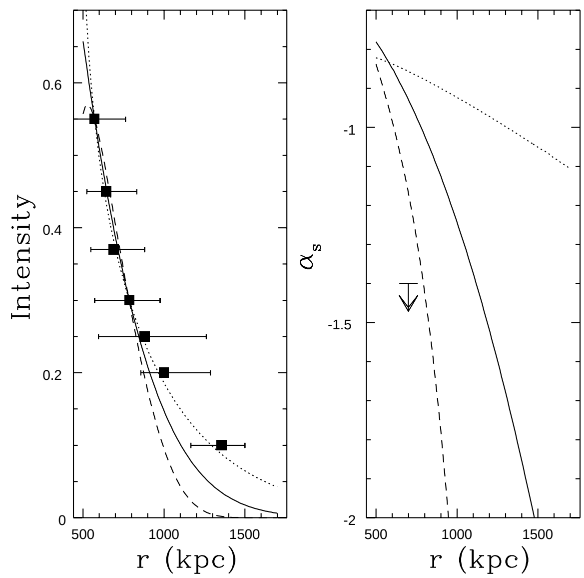

Fig. 4. The one-dimensional surface brightness profiles for the three types of transport $\alpha = 0.5$, 1 and 1.5 fitted to observations of the diffuse radio emission of the Coma cluster of galaxies by Hanisch (1980) as presented by Valtoja (1984). Only the outer parts of the cluster are plotted. Also shown are the predicted dependences of the spectral index on radius. The upper limit is a rough indication of the spectral softening observed by Giovannini et al. (1993)

The conclusion that anomalous transport is necessary is based on assuming that both the transport coefficients and the magnetic field (and also the type of transport as determined by α) are not functions of position, and further, that the transport is independent of particle energy. Although it is not straightforward to include such effects in a transport theory, they do not provide a simple explanation of the observations shown in Fig. 4 in terms of standard diffusive transport. For example, a magnetic field which decreases with distance from the core, could explain the fall-off in the intensity, but could not account for the observed softening of the spectrum, since energy losses would then be less important at larger radius than assumed in the figure. Similarly, if the spatial diffusion coefficient increases in proportion to particle energy – as expected in the case of gyro-Bohm diffusion – then particles responsible for higher frequency radiation are more mobile than those radiating at lower frequency. In this case, the intensity should fall off more slowly at higher frequency,

which would tend to produce a harder spectrum at larger radius, contrary to the observations.

5. Conclusions

In this paper we have presented a simple one-parameter method of modelling the effects of anomalous transport on energetic electrons. Anomalous transport is likely to occur wherever magnetised electrons (i.e., electrons which are tied to field lines) move in a magnetic field which has a stochastic component. In astrophysics, this is the rule rather than the exception and anomalous transport can be expected, for example, in the interstellar medium, as well as in the intra-cluster medium of clusters of galaxies. The primary observational diagnostic is the intensity and spectrum of the synchrotron radiation emitted by the transported particles. For the simplest case in which the magnetic field is of constant magnitude and of random orientation, we have presented general expressions for the surface brightness as a function from position of injection, and also for the spatial variation of the spectral index.

Applying these to the diffuse emission observed from the outer parts of the Coma cluster, we note that it is not possible to distinguish between the various forms of transport merely from the profile of the surface brightness. However, the expected spatial dependence of the spectral index is very sensitive to the type of transport. Assuming that the effects of particle acceleration are negligible in the outer parts of the cluster, and that the electron distribution has achieved a steady state, we find that standard diffusive transport cannot produce the observed rapid softening of the spectrum with radius. Under these assumptions, the type of transport indicated is supra-diffusion, in which particles move almost ballistically in a field configuration which has an ordered radial component.

The computations we have presented contain several major simplifications. In addition to the assumption of constant, randomly orientated magnetic field, and the simple planar or spherical geometry, we have assumed that the parameters governing the transport are independent of the particle's energy. In reality, the type of transport itself (i.e., the value of α) will change depending on the energy range and timescales considered. Thus, at very large times (which may exceed the synchrotron lifetime), a particle can be expected to decorrelate from the magnetic field and perform diffusion (e.g., Duffy et al., 1995). We do not model the situation in which a significant change in α occurs within the synchrotron lifetime of an electron. Finally, in order to apply such models to well-observed objects such as spiral galaxies, it will be necessary to include additional effects such as that of a galactic wind, as well as bremsstrahlung and ionisation loss processes. However, these processes will not change our basic conclusion that it is the spatial dependence of the synchrotron spectral index which provides the most sensitive measure of the transport properties of the emitting electrons.

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A. Derivation of the propagators

The propagators are evaluated by inserting the expansions (13) into Eq. (10), inverting the Fourier transformation analytically (since we fix $\beta = 2$) and using the method of steepest descents to find the asymptotic dependence of the $\Xi(\xi)$ at large ξ. For the one-dimensional case, a standard integral leads to

$$P(x, s) \;=\; \frac{\xi}{s} \exp\left[-\xi (s\,t)^{\alpha/2}\right] \tag{A1}$$

with $\xi = \sqrt{2}(x/\sigma)(\tau_D/t)^{\alpha/2}$, from which, by the method of steepest descents (e.g., Mathews & Walker 1970) one finds

$$\begin{aligned} P(x,t) \;\approx\; & \frac{1}{\sigma\sqrt{2\pi(2-\alpha)}} \left(\frac{\tau_D}{t}\right)^{\alpha/2} \left(\frac{\alpha\xi}{2}\right)^{-(1-\alpha)/(2-\alpha)} \\ & \exp\left[-\left(\frac{2-\alpha}{\alpha}\right)\left(\frac{\alpha\xi}{2}\right)^{2/(2-\alpha)}\right] \end{aligned} \tag{A2}$$

In the case of diffusion ($\alpha = 1$), this result is correctly normalised, i.e., $\int_{-\infty}^{\infty} \mathrm{d}x\, P(x,t) = 1$. However, in general the normalisation is lost in this approach (cf. Rax & White 1992). Correcting for this, we find for the constants used in Eq. (17) the expressions

$$A \;=\; \left(\frac{2-\alpha}{\alpha}\right)\left(\frac{\alpha}{2}\right)^{2/(2-\alpha)} \tag{A3}$$

$$C_d \;=\; \frac{1}{2-\alpha}\left[\frac{d(2-\alpha)}{\pi}\left(\frac{\alpha}{2}\right)^{\alpha/(2-\alpha)}\right]^{d/2} \tag{A4}$$

B. The synchrotron brightness distribution

The expression for the density (26) can be directly integrated numerically. Alternatively, it is easily computed by expanding the factor $(1 - g\gamma t')^{p-2}$ and integrating over t', which yields

$$\begin{aligned} n(\boldsymbol{x}, \gamma) \;=\; & \frac{Q_0 \gamma^{-p-1}}{\alpha \pi^{d/2} g |\boldsymbol{x}|^d} \sum_{k=0}^{+\infty} h_k(p-2) \left(\frac{|\boldsymbol{x}|}{R(\gamma)}\right)^{\frac{2}{\alpha}(k+1)} \\ & \times \Gamma\left(\frac{d}{2} - (k+1)\frac{2-\alpha}{\alpha},\; \left(\frac{|\boldsymbol{x}|}{R(\gamma)}\right)^{\frac{2}{2-\alpha}}\right) \end{aligned} \tag{B1}$$

where

$$R(\gamma) = \frac{2x_s}{\alpha\sqrt{d}}\left(\frac{\alpha}{2-\alpha}\right)^{\frac{2-\alpha}{2}}$$

and

$$h_0(n) = 1\ ,\quad h_k(n) = \frac{(-1)^k}{k!} n(n-1)...(p-k+1)\ .$$

Note that for p integer and larger than 2, the series reduces to $p-2$ terms.

This expression recovers, for 3-dimensional diffusion ($\alpha = 1$, $d = 3$), the formula derived by Wilson (1975), and used by Valtaoja (1984)

$$n(r,\gamma) \;=\; \frac{Q_0\sqrt{g}\Gamma(p-1)\gamma^{-p+1/2}}{8\pi^{3/2}D^{3/2}} \exp\left(-\frac{g\gamma r^2}{4D}\right) U\left(p-1;\frac{3}{2};\frac{g\gamma r^2}{4D}\right) \;, \tag{B2}$$

where U is the confluent hypergeometric function of the second kind (Abramowitz and Stegun 1972). This can be seen either directly from Eq. (26) or, in the special case $p = 2$, by noting that

$$\sqrt{X}e^{-X}U\left(1;\frac{3}{2};X\right) \;=\; \Gamma\left(\frac{1}{2},X\right) \;. \tag{B3}$$

References

Abramowitz M, Stegun I. 1972 ‘Handbook of Mathematical Functions’ National Bureau of Standards, Washington DC.
Achterberg A., Ball L.T. 1994 A&A 285, 687
Balescu R. 1995 Phys. Rev. E 51, 4807.
Berezinskii V.S., Bulanov S.V., Dogiel V.A., Ginzburg V.L., Ptuskin V.S. 1990 ‘Astrophysics of Cosmic Rays’ North-Holland, Amsterdam
Bloemen J.B.G.M., Dogiel V.A., Dorman V.L., Ptuskin V.S. 1993 A&A 267, 372
Bouchaud J.-P., Georges A. 1990 Physics Reports 195, 127
Chuvilgin L.G., Ptuskin V.S. 1993 A&A 279, 278
Crusius A., Schlickeiser R. 1988 A&A 196, 327
Dendy R.O., Duffy P., Gallant Y. A., Kirk J. G. 1995 Proceedings of the 24th. International Cosmic Ray Conference, Rome, 3, 221
Duffy P., Kirk J.G., Gallant Y.A., Dendy R.O. 1995 A&A 302, L21
Getmantsev G.G. 1963 Sov. Astron. J. 6, 477
Giovannini G., Feretti L., Venturi T., Kim K.T., Kronberg P.P. 1993 ApJ 406, 399
Hanisch R.J. 1980 Astron.J. 85, 1565
Hummel E., Dahlem M., van der Hulst J.M., Sukumar S. 1991 A&A 246, 10
Isichenko M.B. 1991a Plasma Physics and Controlled Fusion. 33, 795
Isichenko M.B. 1991b Plasma Physics and Controlled Fusion. 33, 809
Jokipii J.R. 1973 ApJ 183, 1029
Jokipii J.R., Parker E.N. 1969 ApJ 155, 777
Kadomtsev B.B., Pogutse O.P. 1979 Nuclear Fusion Suppl. 1, 649
Kirk J.G., Duffy P., Gallant Y.A. 1996a A&A 314, 1010
Kirk J.G., Duffy P., Dendy R.O. 1996b in ‘Extragalactic Radio Sources, Proceedings of IAU Symposium #175’ Eds. R. Ekers, C. Fanti, L. Padrelli, p. 339, Kluwer, Dordrecht
Lerche I. Schlickeiser R. 1982 MNRAS 201, 1041
Mastichiadis A., Kirk J.G. 1995 A&A 295, 613
Mathews J., Walker R.L. 1970 ‘Mathematical Methods of Physics’ Benjamin/Cummings, London
Montroll E.W., Weiss G.H. 1965 J. Math. Phys. 6, 167
Rax J.M., White R.B. 1992 Phys. Rev. Letts. 68 1523
Rechester A.B., Rosenbluth M.N. 1978 Phys. Rev. Letts. 40 38
Shlesinger M.F., Zaslavsky G.M., Klafter J. 1993 Nature 363, 31
Schlickeiser R., Sievers A., Thiemann H. 1987 A&A 182, 21
Valtaoja E. 1984 A&A 135, 141
Wang H.-D., Vlad M., Vanden Eijnden E., Spineanu F., Misguich J.H., Balescu R. 1995 Phys. Rev. E 51, 4844
Webber W.R., Lee M.A., Gupta M. 1992 ApJ 390, 96
White R.B., Rax J.M., Wu Y. 1993 in ‘Statistical Description of Transport in Plasma, Astro/ and Nuclear Physics’ editors: J. Misguich, G. Pelletier, P. Shuck., Nova Science Publishers, New York, p. 75
Wilson A.S. 1975 A&A 43, 1
Zweibel E.G., Heiles C. 1997 Nature 385, 131

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